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A virial theorem for vortices in ferromagnetic elements

S Komineas

Theory of Condensed Matter Group, Cavendish Laboratory, J J Thomson Avenue, Cambridge CB3 0HE, UK

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Abstract
We derive a virial theorem for a disc-shaped ferromagnetic particle with an axially symmetric magnetic configuration. This is a generalization of Derrick's scaling theorem which is now valid in the presence of surfaces. The relation gives simple results when applied to elementary magnetic states such as a single domain in an infinitely elongated cylinder. We further calculate the vortex state and verify numerically that it satisfies the virial relation. The vortex profile has a simple form in the limit of a very thin particle where also the virial relation simplifies and effectively gives the vortex core radius. Away from the very thin limit, the vortex configuration becomes more complicated with a varying vortex core radius along the thickness of the particle.

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1. Introduction

The fabrication of mesoscopic ferromagnetic elements (henceforth called magnetic elements or magnetic particles) with submicron sizes has made possible a large number of experiments which investigate fundamental as well as technologically important properties of magnetic materials. The small size of these elements allows for a study elementary processes rather than averaged quantities usually measured in bulk magnets. The domain structure is an important magnetic property and there are usually only a few possibilities in small elements. A vortex is a relatively simple nontrivial magnetic state which is often observed in disc-shaped particles with no or little anisotropy [1]. In ring-shaped particles, the vortex takes a particularly simple form [2]. For materials with significant perpendicular anisotropy, a 'magnetic bubble' is the elementary bidomain state and it is observed in particles of appropriate size [3–5]. It is interesting to note that all the above examples refer to axially symmetric magnetic configurations.

The existence of sufficiently simple states, such as those mentioned in the previous paragraph, indicates that there is a possibility for detailed theoretical description of the system, unlike the usual situation in bulk magnets. While exact solutions can rarely be found even in simple cases, analytical work can be carried out in some limits, e.g., in the very thin limit

[6–8], thanks to scaling calculations. The existence of vortices in the same limit has also been studied [9]. The statics and dynamics of the magnetization is governed by the Landau–Lifshitz equation (LLE) which includes various magnetic interactions. The magnetostatic interaction arising from the dipole field of the magnetic moments enters as a nonlocal term in the equation and is of paramount importance here due to the effect of the surfaces of the magnetic element.

We derive here virial relations which should be satisfied by all static solutions in a magnetic particle. Our derivation applies to a disc-shaped particle with an axially symmetric magnetic configuration. Our main result is a Derrick-like relation effectively relying on scaling arguments. The well-known Derrick scaling theorem [10] applies to an infinite ferromagnet without boundaries, while the present virial relation is a significant generalization in that we have included the effect of the surfaces, thus rendering Derrick's theorem applicable in small particles.

The virial relation does not give very specific information in the general case; it can, however, be used as a nontrivial test for solutions that may be found numerically. On the other hand, specific information can be extracted in limiting cases under certain assumptions. One such case is a very thin particle limit. We calculate numerically the vortex close to the limit and verify the virial relation. We also study the vortex in detail as a function of the particle thickness. We study both its profile and its magnetostatic field.

In section 2, we give the derivation of the virial relations and apply the Derrick-like relation to an elementary case. In section 3, we find numerically the vortex solution in a particle and we study the very thin limit. Section 4 contains our conclusions.

2. Virial relations

The energy of a magnetic material can be written as the sum

$$W = W_e + W_a + W_m + W_{\mathrm{ext}}, \tag{1}$$

where we have included the exchange, magnetocrystalline anisotropy, magnetostatic and external field energy, respectively. In rationalized units, these have the form (see e.g. [11, 13])

$$\begin{aligned}
W_e &= \int w_e \,\mathrm{d}V, \qquad w_e = \frac{1}{2}\partial_i \boldsymbol{m}\cdot\partial_i \boldsymbol{m},\\
W_a &= \int w_a \,\mathrm{d}V,\\
W_{\mathrm{ext}} &= \int w_{\mathrm{ext}} \,\mathrm{d}V, \qquad w_{\mathrm{ext}} = -\boldsymbol{h}_{\mathrm{ext}}\cdot\boldsymbol{m},\\
W_m &= -\frac{1}{2}\int \boldsymbol{h}\cdot\boldsymbol{m}\,\mathrm{d}V = \frac{1}{2}\int_{\text{all space}} \boldsymbol{h}^2\,\mathrm{d}V,
\end{aligned} \tag{2}$$

where w_e, w_a and w_{ext} denote the corresponding energy densities. The magnetization vector $\boldsymbol{m} = (m_1, m_2, m_3)$ is supposed to have a constant magnitude which has been normalized to unity $\boldsymbol{m}^2 = 1$. The exchange length has been used as the unit of length. The integrations extend over the particle volume, but the last integration for W_m extends over the entire three-dimensional space.

The anisotropy energy depends on the material, but usual choices are the expressions

$$w_a = \tfrac{1}{2}\left(m_1^2 + m_2^2\right) \qquad \text{and} \qquad w_a = \tfrac{1}{2}m_3^2, \tag{3}$$

for easy-axis and easy-plane anisotropy energy densities, respectively. The magnetostatic field $\boldsymbol{h}$ has been normalized to the saturation magnetization and it satisfies Maxwell's equations $\nabla\times\boldsymbol{h} = 0$, $\nabla\cdot\boldsymbol{b} = 0$, where $\boldsymbol{b} \equiv \boldsymbol{h} + \boldsymbol{m}$ is the magnetic induction used here mostly as a notational abbreviation. The first form of the magnetostatic energy, given in equation (2), is convenient for calculations and the second form shows explicitly that this is positive definite. We have also allowed for an external field $\boldsymbol{h}_{\text{ext}}$ (normalized to the saturation magnetization) which will be assumed to be a uniform one (and time-independent) throughout this paper.

For a magnetic particle with a surface S, the variational problem for the minimization of the energy (1) will be supplemented with the 'unpinned' boundary conditions [11]:

$$\frac{\partial \boldsymbol{m}}{\partial n} = 0, \tag{4}$$

that is, the derivative of the magnetization perpendicular to S vanishes.

In the following, we shall explore in detail only axially symmetric magnetic states of the form

$$m_1 + \mathrm{i}m_2 = [m_\rho(\rho, z) + \mathrm{i}m_\phi(\rho, z)]\,\mathrm{e}^{\mathrm{i}\phi}, \qquad m_3 = m_z(\rho, z), \tag{5}$$

where m_ρ, m_ϕ, m_z are the components of $\boldsymbol{m}$ in cylindrical coordinates, supposed to be functions of ρ and z only. Consequently, the magnetostatic field has no ϕ-component:

$$h_1 + \mathrm{i}h_2 = h_\rho(\rho, z)\,\mathrm{e}^{\mathrm{i}\phi}, \qquad h_3 = h_z(\rho, z). \tag{6}$$

Also, the exchange energy has the form

$$w_e = \frac{1}{2}\left[\left(\frac{\partial \boldsymbol{m}}{\partial \rho}\right)^2 + \frac{m_\rho^2 + m_\phi^2}{\rho^2} + \left(\frac{\partial \boldsymbol{m}}{\partial z}\right)^2\right]. \tag{7}$$

In a first step towards obtaining virial relations one should note that, using formal arguments, it can be shown that static solutions of the equation satisfy the continuity equation [12]

$$\partial_l \sigma_{kl} = 0, \tag{8}$$

where the indices take three values $k, l = 1, 2, 3$ for the three space dimensions x_1, x_2, x_3, and the Einstein summation convention has been adopted. The tensor σ_{kl} can be shown to have the form [12, 13]

$$\sigma_{kl} = \sigma_{kl}^e + \sigma_{kl}^a + \sigma_{kl}^{\text{ext}} + \sigma_{kl}^m, \tag{9}$$

where

$$\begin{aligned} \sigma_{kl}^e &= w_e \delta_{kl} - \partial_k \boldsymbol{m} \cdot \partial_l \boldsymbol{m}, \qquad & \sigma_{kl}^a &= w_a \delta_{kl}, \\ \sigma_{kl}^{\text{ext}} &= w_{\text{ext}} \delta_{kl}, & \sigma_{kl}^m &= h_k b_l - \tfrac{1}{2}\boldsymbol{b}^2 \delta_{kl}. \end{aligned} \tag{10}$$

The form of σ and its elements are given explicitly in the appendix for the case of axial symmetry (this form has also been used in [13]).

We now consider a disc-shaped particle with radius R and thickness t whose axis of symmetry is the z-axis in cylindrical coordinates (ρ, ϕ, z). We denote the top, bottom and side disc surfaces by

$$\begin{aligned} S_+ &\equiv \{z = t/2, \rho \leqslant R\}, \\ S_- &\equiv \{z = -t/2, \rho \leqslant R\}, \\ S_R &\equiv \{\rho = R, -t/2 \leqslant z \leqslant t/2\}, \end{aligned} \tag{11}$$

respectively. An interesting relation is obtained when we integrate the continuity equation (8) over a volume contained in the particle and apply the divergence theorem. We choose a disc-shaped volume with surfaces

$$S_1 \equiv \{z = z_1, \rho \leqslant R\}, \qquad S_2 \equiv \{z = z_2, \rho \leqslant R\}, \qquad S_R' \equiv \{z_1 \leqslant z \leqslant z_2, \rho = R\},$$

where z_1, z_2 are constants with $-t/2 \leqslant z_1 < z_2 \leqslant t/2$. We obtain

$$\int_{S_2} \sigma_{k3}\,\mathrm{d}x_1\,\mathrm{d}x_2 - \int_{S_1} \sigma_{k3}\,\mathrm{d}x_1\,\mathrm{d}x_2 + \int_{S'_R} \sigma_{kl}\,\mathrm{d}S_l = 0, \qquad k = 1, 2, 3. \tag{12}$$

Using (A.1) we find that all three integrals vanish when $k = 1, 2$. For $k = 3$ we obtain

$$\begin{aligned} &\int_{S_2} A 2\pi\rho\,\mathrm{d}\rho - \int_{S_1} A 2\pi\rho\,\mathrm{d}\rho = -2\pi R \int_{S'_R} h_z b_\rho\,\mathrm{d}z, \\ &A \equiv w_e - \left(\frac{\partial \boldsymbol{m}}{\partial z}\right)^2 + w_a + w_{\mathrm{ext}} - \frac{1}{2}\left(2h_\rho m_\rho + h_\rho^2 - h_z^2\right), \end{aligned} \tag{13}$$

where the boundary condition (4) has been employed.

Equation (13) has been arranged so that the lhs is the difference in a form of the energy densities between two slices in the disc particle along its symmetry axis z. It is clear that the energy density varies along the disc thickness due to the presence of the magnetostatic field. On the other hand, for $\boldsymbol{h} = 0$ we would not expect any variation of $\boldsymbol{m}$ with z and (13) would be trivially satisfied.

Further virial relations can be obtained by taking moments of (8) and integrating:

$$\int x_j \partial_l \sigma_{kl}\,\mathrm{d}V = 0.$$

Applying a partial integration and the divergence theorem, we obtain

$$\int_V \sigma_{ij}\,\mathrm{d}V = \oint_S x_j \sigma_{il}\,\mathrm{d}S_l, \tag{14}$$

where S is the boundary of the volume V. When V is the particle volume, we obtain

$$\int_V \sigma_{ij}\,\mathrm{d}V = \int_{S_+} x_j \sigma_{i3}\,\mathrm{d}x_1\,\mathrm{d}x_2 - \int_{S_-} x_j \sigma_{i3}\,\mathrm{d}x_1\,\mathrm{d}x_2 + \int_{S_R} x_j \sigma_{il}\,\mathrm{d}S_l. \tag{15}$$

For an axially symmetric magnetic state of the form (5), equation (15) can conveniently be written as

$$\int_V \sigma_{ij}\,\mathrm{d}V = \int_{S_+} x_j \sigma_{i3} \rho\,\mathrm{d}\rho\,\mathrm{d}\phi - \int_{S_-} x_j \sigma_{i3} \rho\,\mathrm{d}\rho\,\mathrm{d}\phi + \int_{S_R} x_j (\sigma_{i1}\cos\phi + \sigma_{i2}\sin\phi) R\,\mathrm{d}\phi\,\mathrm{d}z. \tag{16}$$

We first consider the case $i, j = 1, 2$. We apply (14) separately in the interior of the particle and in the space outside the particle thus obtaining two equations. We take into account that the magnetostatic field ($\boldsymbol{h}$) component parallel to the particle surface and the magnetic induction ($\boldsymbol{b}$) component perpendicular to the particle surface are continuous. Combining the two relations, we obtain

$$\int_{\text{all space}} \sigma_{11}\,\mathrm{d}V = \int_{\text{all space}} \sigma_{22}\,\mathrm{d}V = \pi R^2 \int_{S_R} \left[w_e + w_a + w_{\mathrm{ext}} - \boldsymbol{h}\cdot\boldsymbol{m} - \frac{1}{2}\left(1 + m_\rho^2\right)\right] \mathrm{d}z. \tag{17}$$

It is understood that the surface integral over S_R has been reduced to a simple integration in z over the interval $[-t/2, t/2]$. We then consider $i = j = 3$ in equation (14) and follow the same steps as above to obtain

$$\begin{aligned} \int_{\text{all space}} \sigma_{33}\,\mathrm{d}V = &\frac{t}{2} \int_{S_+} \left[w_e + w_a + w_{\mathrm{ext}} - \boldsymbol{h}\cdot\boldsymbol{m} - \frac{1}{2}\left(\boldsymbol{m}^2 + m_z^2\right)\right] 2\pi\rho\,\mathrm{d}\rho \\ &+ \frac{t}{2} \int_{S_-} \left[w_e + w_a + w_{\mathrm{ext}} - \boldsymbol{h}\cdot\boldsymbol{m} - \frac{1}{2}\left(\boldsymbol{m}^2 + m_z^2\right)\right] 2\pi\rho\,\mathrm{d}\rho. \end{aligned} \tag{18}$$

It is understood that the surface integrals over $S_\pm$ have been reduced to simple integrations of ρ over the interval [0, R].

We now use equations (9), (10) to obtain

$$\mathrm{Tr}\sigma \equiv \sigma_{11} + \sigma_{22} + \sigma_{33} = w_e + 3[w_a + w_{\mathrm{ext}}] - \tfrac{1}{2}\boldsymbol{h}^2 - 2\boldsymbol{h}\cdot\boldsymbol{m} - \tfrac{3}{2}. \tag{19}$$

Integrating this over all space and using equations (2), (17), (18), we obtain the Derrick-like relation

$$\begin{aligned} W_e + 3(W_a + W_{\mathrm{ext}} + W_m) = 2\pi R^2 \int_{S_R} &\left[w_e + w_a + w_{\mathrm{ext}} - \boldsymbol{h}\cdot\boldsymbol{m} - \frac{1}{2} m_\rho^2 \right] \mathrm{d}z \\ + \frac{t}{2} \int_{S_+} &\left[w_e + w_a + w_{\mathrm{ext}} - \boldsymbol{h}\cdot\boldsymbol{m} - \frac{1}{2} m_z^2 \right] 2\pi\rho\, \mathrm{d}\rho \\ + \frac{t}{2} \int_{S_-} &\left[w_e + w_a + w_{\mathrm{ext}} - \boldsymbol{h}\cdot\boldsymbol{m} - \frac{1}{2} m_z^2 \right] 2\pi\rho\, \mathrm{d}\rho. \end{aligned} \tag{20}$$

The standard Derrick relation [10] applies to an infinite ferromagnet without boundaries and it is readily obtained from (20) by setting all surface integrals to zero (that is the rhs of the equation would vanish). In the absence of an external field ($\boldsymbol{h}_{\mathrm{ext}}$), the lhs of equation (20) is manifestly positive definite and thus the standard Derrick relation would exclude any static nontrivial solutions in a three-dimensional ferromagnet without boundaries. The surfaces are responsible for rendering this conclusion invalid in a ferromagnetic element as acknowledged by the generalized Derrick form (20). The latter equation does not exclude nontrivial static solutions, which may give a positive surface integral on the rhs.

An application of equation (20) to elementary cases might appear straightforward. We shall study here the case of an infinitely elongated cylinder and an isotropic material ($w_a = 0$). We consider a disc whose thickness $t \to \infty$ while its radius R remains fixed. We find numerically that, in this limit, the magnetization as a function of the distance from the surfaces $S_\pm$ reaches a limit. In the disc centre, away from $S_\pm$, this is uniform and takes one of the values $\boldsymbol{m} = (0, 0, \pm 1)$ so that it is parallel to the side surface of the magnetic element in the bulk of the cylinder. Consequently, all energy terms and the integral over S_R in (20) have finite values as $t \to \infty$ since $\boldsymbol{m}$ approaches the uniform value fast enough, as confirmed by numerical results. This necessarily implies that the integrals over $S_\pm$ should vanish as $1/t$ or faster so that the rhs remains finite in the limit $t \to \infty$. Therefore, we have

$$\int_{S_\pm} \left[w_e - \boldsymbol{h}\cdot\boldsymbol{m} - \frac{1}{2} m_z^2 \right] \rho\, \mathrm{d}\rho = 0 \quad (\text{for} \quad t = \infty), \tag{21}$$

where the notation $S_\pm$ implies either of the surfaces S_+ or S_-. We actually find numerically that the second and third terms on the rhs of equation (20) take nonzero values in the limit $t \to \infty$.

3. A vortex in a particle

Vortices are among the most prominent examples of magnetic states which have been studied theoretically for a long time in the context of two-dimensional models. However, related experiments have been rare. The situation has been reversed in the last decade due to a large number of experiments in magnetic elements where various vortex states have been observed. An axially symmetric vortex appears to be among the most important magnetic configurations most notably in disc- and ring-shaped particles of materials with little or no anisotropy [1, 2]. Observations of the details of a vortex core in magnetic particles have been reported in [14, 15].

We consider here a disc-shaped particle with no anisotropy $w_a = 0$ (also consider $w_{\rm ext} = 0$). We study an axially symmetric vortex which has the form (5) where $m_z(\rho = 0, z) = \pm 1$ while $m_z(\rho = R, z) \approx 0$. The third component of the magnetization m_z vanishes outside a region which is called the vortex core and this is usually tacitly assumed to be independent of z. One supposes $m_\rho \approx 0, m_\phi \approx \pm 1$ outside the vortex core so that there are no surface charges on S_R. In fact, this choice means that away from the vortex core the magnetostatic field vanishes ($\boldsymbol{h} = 0$). On the other hand, the exchange energy grows logarithmically with the particle radius R. In conclusion, we have four kinds of axially symmetric vortices depending on the value of m_z in the vortex centre and also on the value of m_ϕ away from the vortex core. These are related by simple symmetry transformations and they have the same energy.

We apply the virial relation (13) for a vortex in a particle with radius R much larger than the vortex core radius. Since $\boldsymbol{h} = 0$ on S_R, equation (13) states that the quantity

$$\int_0^R \left[w_e - \left(\frac{\partial \boldsymbol{m}}{\partial z}\right)^2 - \frac{1}{2}\left(2h_\rho m_\rho + h_\rho^2 - h_z^2\right)\right] 2\pi\rho\, \mathrm{d}\rho \tag{22}$$

is independent of z. The same relation has been derived in [13] for a localized soliton in an infinite film.

In numerical calculations, we find that the vortex profile satisfies the parity relations

$$m_\rho(\rho, z) = -m_\rho(\rho, -z), \qquad m_\phi(\rho, z) = m_\phi(\rho, -z), \qquad m_z(\rho, z) = m_z(\rho, -z), \tag{23}$$

which are compatible with the Landau–Lifshitz equation of motion. Corresponding parity relations are implied for the magnetostatic field: $h_\rho(\rho, z) = -h_\rho(\rho, -z), h_z(\rho, z) = h_z(\rho, -z)$. Using the above parity relations and confining ourselves to the case of vanishing anisotropy and no external field, the virial relation (20) assumes the simpler form

$$W_e + 3W_m = 2\pi R^2 \int_{S_R} \left[w_e - \boldsymbol{h}\cdot\boldsymbol{m} - \frac{1}{2}m_\rho^2\right] \mathrm{d}z + t\int_{S_\pm}\left[w_e - \boldsymbol{h}\cdot\boldsymbol{m} - \frac{1}{2}m_z^2\right] 2\pi\rho\,\mathrm{d}\rho, \tag{24}$$

where $S_\pm$ means that either S_+ or S_- may be used. In the case of a particle radius R much larger than the vortex core radius, where $\boldsymbol{h} = 0, m_\rho = m_z = 0, m_\phi = \pm 1$ on S_R, we find $w_e = 1/2R^2$ on S_R provided $\boldsymbol{h}$ and $\boldsymbol{m}$ fall fast enough with ρ. Then equation (24) reduces to

$$W_e + 3W_m = t\pi + t\int_{S_\pm}\left[w_e - \boldsymbol{h}\cdot\boldsymbol{m} - \frac{1}{2}m_z^2\right] 2\pi\rho\,\mathrm{d}\rho, \tag{25}$$

which states that bulk properties are reflected on the particle surface $S_\pm$.

More specific results for the vortex can be derived in the limit of a very thin particle. The results of [6] (extended to relative minimizers) suggest that, in the limit $t \to 0$, the magnetization $\boldsymbol{m}$ does not depend on z and $h_\rho = 0, h_z = -m_z$. Employing further the assumptions $m_\phi = \pm 1$, $w_e = 1/2R^2$ on S_R, equation (24) reduces to

$$\int m_z^2 2\pi\rho\,\mathrm{d}\rho = \pi. \tag{26}$$

The same relation has been derived in the appendix of [16], where the vortex in a two-dimensional easy-plane magnet (without a magnetostatic field) was studied. The radius of the

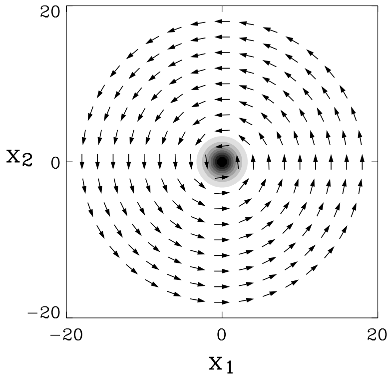

Figure 1. The vortex in a disc-shaped particle with radius $R = 19.9$ and thickness $t = 3.8$. We show the top particle surface but there is little variation of the profile along the particle thickness. The arrows show the projection of the magnetization vector $\boldsymbol{m}$ on the (x_1, x_2) plane while the third component of the magnetization (m_z) is represented in grey scale. Black corresponds to maximum length of $m_z = -1$ and white corresponds to $m_z = 0$ ($m_z \to 1$ is also a). The vortex profile may be fitted (albeit poorly) by (27) with $\rho_0 = 1.2$. In this and following figures, length is measured in exchange length units.

vortex core can be estimated if we use the model

$$m_\rho = 0, \qquad m_\phi = \tanh(\rho/\rho_0), \qquad m_z = \frac{1}{\cosh(\rho/\rho_0)} \tag{27}$$

and substitute in (26) to obtain $\rho_0 = 0.85$.

We now proceed to a numerical calculation of the vortex for varying particle thickness. Our code [13] solves the Landau–Lifshitz–Gilbert equation in order to find static configurations. We use finite differences to discretize space in cylindrical coordinates (ρ, z). The magnetostatic field $\boldsymbol{h}$ is calculated by a combination of the conjugate gradient method and a direct integration of the Poisson equation. We typically use a lattice spacing $\Delta\rho = \Delta z = 0.2$ but we also use smaller values for very thin particles. In all the results, the virial relation (20) is satisfied to an accuracy better than 1%.

We choose a radius $R = 19.9$ for the particle and find numerically the vortex for a range of thicknesses. In figure 1, we present the result for a relatively thin particle through the projection of the magnetization vector on the (x, y) plane while the third component of the magnetization (m_z) is represented in grey scale. m_ϕ, m_z present little variation along the particle thickness (z-axis) and the profile can be fitted by the model (27) with $\rho_0 = 1.2$. However, m_ρ does vary along z and it has some significant value (up to $m_\rho \approx 0.1$) near the surfaces while it vanishes at $z = 0$, thus signalling a departure from the very thin limit.

Experiments usually measure the magnetostatic field produced by the magnetic particle rather than the magnetization itself; thus we present in figure 2 the two nonzero components of $\boldsymbol{h}$ just over the top surface of the particle. The $\boldsymbol{h}$ field has a significant value at the central region which should be attributed to the magnetization being perpendicular to the top and bottom surfaces at the vortex core. The magnetostatic field falls to very small values away from the vortex core which is because the vortex configuration in this region has the form $\boldsymbol{m} = \hat{\phi}$ and is thus solenoidal $(\nabla \cdot \boldsymbol{m} = 0)$ while it produces no surface charges. In the very thin limit, the result in [6] means that the magnetostatic field outside the particle vanishes apparently due to the trivial fact that the total magnetization of the vortex core vanishes for

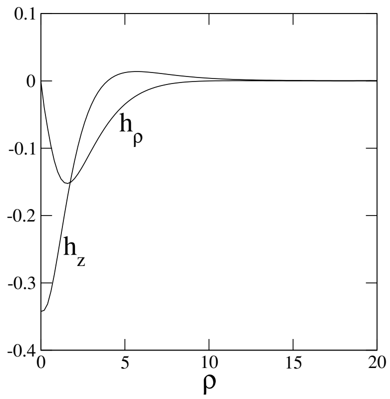

Figure 2. The magnetostatic field for the vortex of figure 1 just over the top surface of the particle. We present the two nonzero components h_ρ, h_z as functions of ρ. In this and following figures, the magnetostatic field is measured in units of the saturation magnetization.

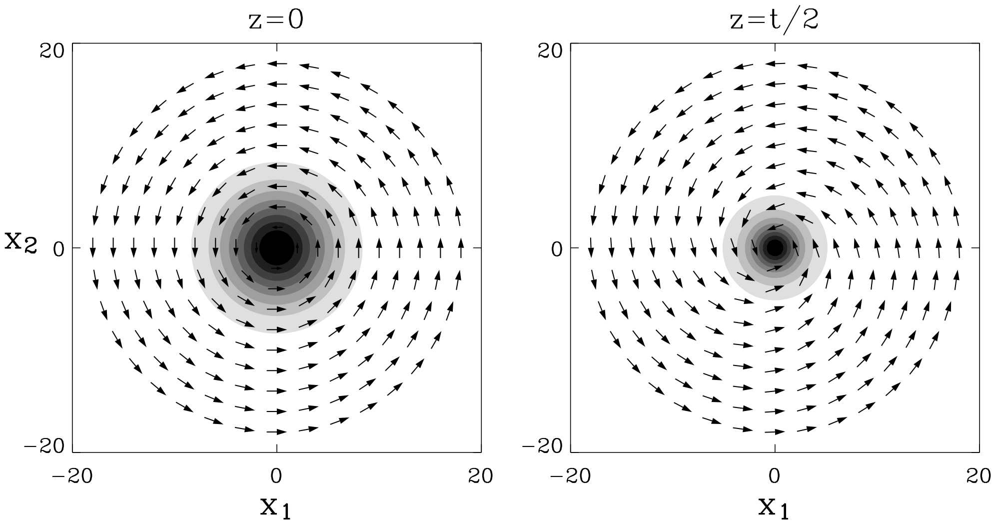

Figure 3. The vortex in a disc-shaped particle with radius $R = 19.9$ and thickness $t = 15.8$. The vortex profile at the left plane ($z = 0$) is fitted (albeit poorly) by (27) with $\rho_0 = 3.0$. At the right surface ($z = t/2$), m_ρ is nonvanishing and the corresponding vortex core radius is $\rho_0 = 1.6$.

$t \to 0$. This is consistent with our numerical results although we have not attempted to explicitly verify it numerically.

It is interesting, also for practical purposes, to find how the vortex profile evolves as the thickness of the particle increases. Thus, we consider a particle with the same radius $R = 19.9$ as in the previous example but with a larger thickness $t = 15.8$. In figure 3, we present the magnetic configuration at the left ($t = 0$) and right planes ($z = t/2$) of the particle. The profile at the bottom surface can be obtained by the parity relations (23). Unlike in very thin particles, the vortex profile now varies significantly across the particle thickness. The vortex core radius has increased compared to that of figure 1 and this is significantly larger in the middle plane than near the top and bottom surfaces. In the top surface ($z = t/2$), it is clear

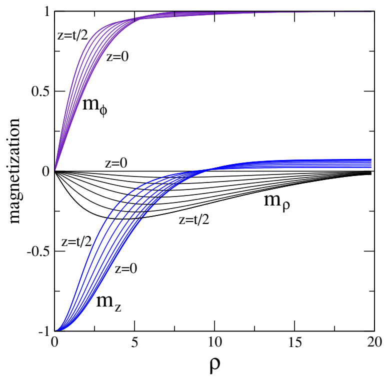

Figure 4. The vortex profile for a particle with radius $R = 19.9$ and thickness $t = 15.8$. The lines correspond to the magnetization components m_ρ, m_ϕ, m_z, as indicated, as functions of the radial coordinate ρ at the levels $z = 0, 1, 2, 3, 4, 5, 6, 7, t/2$.

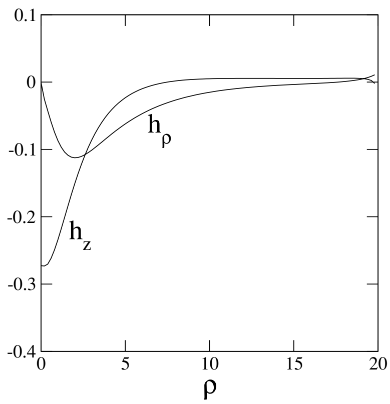

Figure 5. The magnetostatic field for the vortex of figure 3 just over the top surface of the particle. We present the two nonzero components h_ρ, h_z as functions of ρ.

that m_ρ is nonvanishing, unlike in the very thin limit. On the other hand, we find that $m_\rho = 0$ in the middle plane ($z = 0$) as is suggested by (23). We also find that m_z has a significant value for $\rho = R$ near the middle plane ($z = 0$).

A more detailed view of the vortex profile is given in figure 4. We plot the components of $\boldsymbol{m}$ as functions of ρ for various values of z along the particle thickness. The variation of the vortex core radius is probably most apparent in the form of m_z which is broader at $z = 0$. The graph for m_ρ shows that this is significant at the vortex core while it falls to zero very slowly with ρ. We note that the vortex profile remains almost unaltered for radii $R > 20$ with $m_\rho \approx 0, m_\phi \approx 1, m_z \approx 0$ for all ρ large compared to the vortex core radius.

Figure 5 shows the two nonzero components of $\boldsymbol{h}$ just over the top surface of the particle. In comparison to figure 2, the magnetostatic field now falls slower due to the larger size of the vortex core. A remark is in order concerning the magnitude of h_z over the centre of the vortex core ($\rho = 0$). The magnetostatic field should vanish in the very thin limit, as has been discussed earlier, essentially due to the vanishing total magnetization. On the other hand, it appears that $h_z(\rho = 0)$ for $t = 15.8$ in figure 5 is reduced in comparison to the corresponding value for $t = 3.8$ in figure 5. We infer that $h_z(\rho = 0, z = t/2)$ should have an extremum, and we find numerically that this is attained for a thickness close to $t = 3.8$.

We finally address the limit $t \to \infty$ with R kept fixed, that is, that of an elongated pilar. We assume that the magnetization profile in the vicinity of the top and bottom surfaces may be a vortex configuration. Following the discussion at the end of section 2, we assume that $\boldsymbol{m}$ is almost uniform in the bulk of the pilar in order to minimize the exchange and magnetostatic energies. Under these assumptions, equation (21) should be satisfied also in the present case. However, this is a very stringent condition for a vortex state. In order to see this, suppose that the vortex core radius at the surface $S_\pm$ would be small compared to R. Since the region outside the vortex core would give $w_e > 0, \boldsymbol{h} = 0, m_z = 0$ and thus would contribute a positive quantity to the integral in equation (21), the equation would not be satisfied. We conclude that, if a vortex exists at all for $t \to \infty$ at fixed R, then the vortex core radius would have to be comparable to R. Numerical simulations for $R = 5$ have shown that the particle cannot sustain a vortex for large thicknesses. The above considerations certainly do not exclude other solutions which may be similar but probably more complicated than the vortex considered here.

4. Conclusions

We have been motivated by the very extensive experimental activity on mesoscopic ferromagnetic particles of the last years to investigate interesting axially symmetric magnetic configurations such as a vortex. We have derived a relation, (equation (20)) which should be obeyed by all axially symmetric static solutions in a disc-shaped particle. We have further given examples of how this rather general relation can be used to give more specific information in the very thin and very thick particle limits. The most prominent example of an axially symmetric state is a vortex which has been observed in many experiments. We elaborate on the vortex solution and solve the Landau–Lifshitz equation numerically to show that the details of this configuration depend on the particle thickness.

There is currently substantial theoretical interest in the subject of geometrically constrained ferromagnetic bodies motivated largely by the nontrivial effects of the magnetostatic field which is a nonlocal interaction ([6–9] and references therein). We hope that the present results will contribute towards further progress for the study of properties of magnetic states in this new context.

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Appendix. The elements of σ

We use definitions (9), (10), and equations (5), (6) to find that the elements of the tensor σ_{kl}, in the case of axially symmetric solutions, have the form (we suppose $\mu, \nu = 1, 2$)

$$\begin{aligned}
&\sigma_{\mu\nu} = C^{(0)}_{\mu\nu} + C^{(1)}_{\mu\nu}\cos(2\phi) + C^{(2)}_{\mu\nu}\sin(2\phi), \qquad && \sigma_{\mu 3} = C^{(1)}_{\mu 3}\cos\phi + C^{(2)}_{\mu 3}\sin\phi,\\
&\sigma_{33} = C^{(0)}_{33}, && \sigma_{3\nu} = C^{(1)}_{3\nu}\cos\phi + C^{(2)}_{3\nu}\sin\phi,
\end{aligned} \tag{A.1}$$

where the coefficients are functions of ρ and z only, and are given below.

$$
\begin{aligned}
C_{11}^{(0)} &= C_{22}^{(0)} = \frac{1}{2}\left(\frac{\partial \boldsymbol{m}}{\partial z}\right)^2 + w_a + w_{\text{ext}} - \frac{1}{2}\left(1 + h_\rho m_\rho + 2h_z m_z + h_z^2\right),\\
C_{11}^{(1)} &= -C_{22}^{(1)} = C_{12}^{(2)} = C_{21}^{(2)} = -\frac{1}{2}\left[\left(\frac{\partial \boldsymbol{m}}{\partial \rho}\right)^2 - \frac{m_\rho^2 + m_\phi^2}{\rho^2}\right] + \frac{1}{2}\left(h_\rho m_\rho + h_\rho^2\right),\\
C_{11}^{(2)} &= -C_{22}^{(2)} = -C_{12}^{(1)} = -C_{21}^{(1)} = \frac{1}{\rho}\left(m_\rho \frac{\partial m_\phi}{\partial \rho} - m_\phi \frac{\partial m_\rho}{\partial \rho}\right) - \frac{1}{2} h_\rho m_\phi,\\
C_{33}^{(0)} &= w_e - \left(\frac{\partial \boldsymbol{m}}{\partial z}\right)^2 + w_a + w_{\text{ext}} - \frac{1}{2}\left(1 + 2h_\rho m_\rho + h_\rho^2 - h_z^2\right),\\
C_{12}^{(0)} &= -C_{21}^{(0)} = \frac{1}{2} h_\rho m_\phi,\\
C_{13}^{(1)} &= C_{23}^{(2)} = -\frac{\partial \boldsymbol{m}}{\partial \rho}\frac{\partial \boldsymbol{m}}{\partial z} + h_\rho m_z + h_\rho h_z,\\
C_{13}^{(2)} &= -C_{23}^{(1)} = \frac{1}{\rho}\left(m_\rho \frac{\partial m_\phi}{\partial z} - m_\phi \frac{\partial m_\rho}{\partial z}\right),\\
C_{31}^{(1)} &= C_{32}^{(2)} = -\frac{\partial \boldsymbol{m}}{\partial \rho}\frac{\partial \boldsymbol{m}}{\partial z} + h_z m_\rho + h_\rho h_z,\\
C_{31}^{(2)} &= -C_{32}^{(1)} = \frac{1}{\rho}\left(m_\rho \frac{\partial m_\phi}{\partial z} - m_\phi \frac{\partial m_\rho}{\partial z}\right) - h_z m_\phi.
\end{aligned}
\tag{A.2}
$$

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